Journal of Statistical Physics, Vol. 65, Nos. 3/4, 1991

A Maximum-Entropy Principle for Two-Dimensional Perfect Fluid Dynamics

Raoul Robert[1]

Received June 18, 1991

We use Kullback entropy for Young measures to define statistical equilibrium states for a two-dimensional incompressible flow of a perfect fluid. This approach is justified, as it gives a concentration property about the equilibrium state in the phase space. It might give a statistical understanding of the appearance of coherent structures in two-dimensional turbulence.

KEY WORDS: Statistical mechanics of two-dimensional Euler equations; Young measures; Kullback entropy; equilibrium states; coherent structures; turbulence.

1. INTRODUCTION

The appearance of coherent structures is one of the most striking features of two-dimensional turbulence. While there is an obvious tendency in ordinary fluid turbulence for the system to try to increase its disorder, at the same time there are circumstances in which a sort of "macroscopic" order seems to emerge from what appears to be "microscopic" disorder. The question that we address here is: how can this behavior of the fluid be explained or predicted from Euler equations, which govern the dynamics of an incompressible perfect fluid?

The observation of the merging of two like-sign vortices (experimentally or on numerical simulations[15,30]) shows that the final "macroscopic" state does not depend on the very variable nature of the intermediate "microscopic" states, due to the complicated deformation of the vortices by mutual straining. This suggests that an explanation of the phenomenon

[1] 21, Avenue Plaine Fleurie, 38240 Meylan, France.

0022-4715/91/1100-0531$06.50/0 © 1991 Plenum Publishing Corporation

must be of a statistical nature. Of course this conclusion is not new, and there have been several attempts to build up some statistical hydrodynamics, begining with the pioneering work of Onsager.[29]

We can classify all attempts to apply the methods of statistical mechanics to fluid dynamics in two categories. In the first one we find several works which have continued Onsager's approach. The idea was to approximate the continuous Euler system by a great (but finite) number of point vortices. This leads to a finite-dimensional Hamiltonian system, to which can be applied the methods of statistical mechanics (see, for example, refs. 14, 25, 27, 31, and 36). Interesting presentations and discussions of this approach can be found in refs. 25 and 36. Though very enlightening, this approach reveals a severe difficulty. There are many different ways to approximate a continuous vorticity by a cloud of point vortices. And different approximations can lead to very different statistical equilibrium states. So, the thermodynamic equilibrium state that we can associate to a continuous vorticity depends dramatically on arbitrary choices (this difficulty was underlined by Onsager).

There is another way to approximate the two-dimensional Euler system, in the case of spatially periodic flows: we decompose the vorticity into a Fourier serie and truncate the description to a finite number of Fourier coefficients. One can prove a Liouville theorem for the truncated system (in the phase space of Fourier coefficients, the volume element is conserved). This suggests again that the methods of statistical mechanics be employed.[19,21] It happens that after the truncation, only two constants of the motion remain. They are the energy, and the mean square vorticity or enstrophy. Then the Gibbs canonical ensemble corresponding to these two constraints is easily obtained. Here also, there is a serious obstacle. When we consider the truncated system, instead of the full Euler system, we lose the information given by all the integral functionals of the vorticity, which are constants of the motion for the full system (this is due to the law of vorticity conservation along the trajectories of the fluid particles). As a consequence, the significance of the equilibrium states of the truncated system for the full one is far from being obvious. If the system were ergodic in some sense (this is the underlying hypothesis of any statistical mechanics approach), it could only be on the "submanifold" of the phase space defined by all the constants of the motion fixed at their initial value. To overcome this difficulty, one can try to construct Gibbs states for the full Euler system by a limit process when the number of Fourier coefficients goes to infinity. A lot of work has been devoted to the study of probability measures of the Gibbs form with formal densities given by the enstrophy and the energy, and to the problem of the construction of an associated equilibrium dynamics; see, for example, ref. 1 and the references therein.

We notice also the very interesting contribution of ref. 8, where the authors succeeded in constructing a family of Gibbs states associated to the law of vorticity conservation along the trajectories of the fluid particles. Unfortunately, these probability measures are supported by very "large" functional spaces of generalized functions; so that not only are the mean energy and enstrophy of these states infinite, but the phase space of bounded measurable vorticity functions, on which the classical Eulerian flow can be defined, is of null measure. So it is only at a formal level that this make sense.

The main conclusion that can be drawn from this short overview is that, although the finite-dimensional approximations of Euler equations can provide a good representation of the flow during a finite time, the information that the thermodynamics (or long-time dynamics) of such systems gives on the behavior of the full system is highly questionable.

We propose here a new approach to the problem. We work in the infinite-dimensional phase space $L^\infty(\Omega)$ (for the vorticity functions). The Cauchy problem for Euler equations is well posed in that space (we have existence and uniqueness of the solution for all time). But we do not have a Liouville measure, as in the finite-dimensional case. Our main result is that for any given initial vorticity function ω_0, the set of all the vorticity functions ω in $L^\infty(\Omega)$ which allow all the constants of the motion to be equal to their initial values (on ω_0) is concentrated (in a natural sense, closely related to large-deviation theory) about a very particular set, the equilibrium set. To find the equilibrium set, we introduce a macroscopic description of the small-scale oscillations of the vorticity functions by means of Young measures. Then, the maximization of the Kullback entropy of these Young measures (which was introduced in a previous work[32]) enables us to find the equilibrium set. This set is conserved by the Eulerian flow. We write down the equation of Gibbs states which is necessarily satisfied by the functions of the equilibrium set, and we give a condition which ensures that this equation has a unique solution, the equilibrium state.

Besides the fact that we work with the full Euler system and take into account all the known constants of the motion, our approach has the following advantages. First, we can provide mathematical proofs of the essential properties of concentration and invariance. Furthermore, it gives clear formulas that permit quantitative confrontation with experiments[38] and numerical simulations.[41]

2. BASIC RESULTS ON THE TWO-DIMENSIONAL INCOMPRESSIBLE EULER SYSTEM

Throughout this paper we shall work with the solutions of the incompressible Euler system in an open bounded regular domain Ω of the plane. This system is usually written

$$\begin{aligned} &u_t + (u \cdot \nabla) u = -\nabla p \\ &\operatorname{div} u = 0 \\ &u \cdot n = 0 \quad \text{on} \quad \partial\Omega \quad (n \text{ normal to } \partial\Omega) \\ &u(0, x) = u_0(x) \end{aligned}$$

Here $u(t, x)$ is the velocity field of the fluid and p the normalized pressure.

For the sequel, we find it convenient to introduce the scalar vorticity $\omega = \operatorname{curl} u$, and we write the system in the velocity–vorticity formulation:

$$(\mathrm{E}) \quad \begin{cases} \left.\begin{aligned} &\omega_t + \operatorname{div}(\omega u) = 0 \\ &\omega(0, x) = \omega_0(x) \end{aligned}\right\} & (\mathrm{E1}) \\ \left.\begin{aligned} &\operatorname{curl} u = \omega \\ &\operatorname{div} u = 0 \\ &u \cdot n = 0 \quad \text{on} \quad \partial\Omega \end{aligned}\right\} & (\mathrm{E2}) \end{cases}$$

This Euler system appears as a transport equation (E1) coupled with the elliptic system (E2).

The system (E2) is classically solved by introducing the stream function ψ, defined by

$$\begin{aligned} -\Delta\psi &= \omega \\ \psi &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Then $u = \operatorname{curl} \psi$ gives the solution of (E2).

On the other hand, (E1) is solved by introducing the Lagrangian flow φ_t defined by

$$\begin{aligned} \frac{d}{dt} \varphi_t(x) &= u(t, \varphi_t(x)) \\ \varphi_0(x) &= x \end{aligned}$$

One can easily check that the vorticity is convected by the flow, that is, $\omega(t, \varphi_t(x)) = \omega_0(x)$, for all x in Ω.

It is well known[7,18,42] that if ω_0 is a smooth function on $\bar{\Omega}$, (E) has a unique classical solution for all time. Although these classical results do not permit us to take for initial data any ω_0 in the space $L^\infty(\Omega)$, we can define weak solutions (in the distribution sense), and by a limit process we get the following result well suited to our needs.[42]

Theorem 1. For ω_0 in $L^\infty(\Omega)$, we have a unique weak solution of (E), $\omega(t, x)$ in the space $L^\infty(]0, +\infty[\times\Omega)$. Furthermore, $\omega(t, x)$ and the corresponding $u(t, x)$ and $\varphi_t(x)$ satisfy:

(i) $\omega(t, \varphi_t(x)) = \omega_0(x)$, for almost all t, x in $]0, +\infty[\times\Omega$.

(ii) $|u(t, x) - u(s, y)| \leqslant c(\Omega)\|\omega_0\|_\infty[\sigma(|t-s|) + \sigma(|x-y|)]$, where $\sigma(\varepsilon) = \varepsilon(1 + |\mathrm{Log}\,\varepsilon|)$.

We will say that u is quasi-Lipschitz in t and x.

(iii) The mappings φ_t: $\Omega \to \Omega$ are area-preserving homeomorphisms satisfying

$$|\varphi_t(x) - \varphi_t(y)| \leqslant |x-y|^{e^{-Mt}}$$

with $M = 2c(\Omega)\|\omega_0\|_\infty$.

We shall take $L^\infty(\Omega)$ as phase space and define the Eulerian flow Φ_t: $L^\infty \to L^\infty$ by $\Phi_t\omega(x) = \omega(\varphi_t^{-1}(x))$.

One easily checks that the following functionals are conserved by the flow:

(i) The energy $E(\omega) = \frac{1}{2}\int_\Omega u^2\,dx = \frac{1}{2}\int_\Omega \psi\omega\,dx$.

(ii) $\int_\Omega f(\omega(x))\,dx$, for any continuous function f on $\mathbf{R}$.

We clearly define a positive bounded Radon measure π_ω on $\mathbf{R}$ by

$$\langle \pi_\omega, f\rangle = \int_\Omega f(\omega(x))\,dx$$

π_ω is the distribution measure of ω.

So, we can say that π_ω is conserved by the flow, i.e.,

$$\pi_{\Phi_t\omega} = \pi_\omega$$

(iii) If Ω is not simply connected, there are other constants of the motion given by the circulation around the obstacles.

(iv) If Ω is a ball $B(0, R)$, we must take into account the angular momentum with respect to 0:

$$\int_\Omega x \wedge u(x)\,dx = \frac{1}{2}\int_\Omega (R^2 - x^2)\,\omega(x)\,dx$$

and also the two components of the linear momentum $\int_\Omega x\omega(x)\,dx$ in the case $\Omega = \mathbf{R}^2$.

It can be shown[37] that there are not any other invariant functionals of the form $\int_\Omega F(x, u(x), \nabla u(x))\,dx$ than those we already knew.

In the case of stationary solutions, (E) reduces to

$$\operatorname{div}(\omega \operatorname{curl} \psi) = 0$$

This is satisfied, for example, if $\omega = f(\psi)$, where f is a continuous function, or if ω is rotation invariant.

To conclude this short section, let us notice that despite the fact that Euler equations appear as an infinite-dimensional Hamiltonian system (see ref. 3 for the Lagrangian viewpoint and ref. 28 for the Eulerian viewpoint), we do not know how to get something like a Liouville measure on the phase space $L^\infty(\Omega)$.

3. MACROSCOPIC DESCRIPTION BY YOUNG MEASURES

Let us consider the fluid motion corresponding to some initial data $\omega_0(x)$. Denoting $m = \|\omega_0\|_\infty$, we know that, for all t, $\omega(t)$ is in $L^\infty_m = \{\omega \in L^\infty \,\|\, \|\omega\|_\infty \leqslant m\}$.

Let us suppose, for example, that ω_0 consists of a finite number of patches of uniform vorticity. Then we know that the boundaries of the patches become in general more and more intricate as time goes on, but the area of each vorticity patch is conserved, as well as the total kinetic energy of the system. Since the vorticity contours become so intricate, we are not really interested in the exact vorticity field. Indeed the velocity field results from an integration of the vorticity, so that it does not depend on the fine-scale fluctuations of the vorticity: it depends only on its local average. In fact, to exploit the whole information given by the constants of the motion, we are led to consider a macroscopic description of the system by introducing the local probability distribution of the different vorticity levels in a small neighborhood. Therefore, we define a macroscopic state as a field of these local probabilities, while an exact vorticity field is called here a microscopic state.

In other words, we introduce a macroscopic description by immersing the phase space $L^\infty_m(\Omega)$ in the set of Young measures on $\Omega \times [-m, m]$.

Let us recall that Young measures[32,43] are a natural way to generalize the notion of measurable mapping from Ω to $[-m, m]$: at any point $x \in \Omega$, we no longer have a well-determined value, but only some probability distribution on $[-m, m]$. In other words, a Young measure ν is a measurable mapping $x \rightsquigarrow \nu_x$ from Ω to the set $M_1([-m, m])$ of the Radon probability measures on $[-m, m]$ endowed with the narrow topology (weak topology associated to the continuous functions on $[-m, m]$).

Clearly, ν defines a positive Radon measure on $\Omega \times [-m, m]$ (which we will also denote by ν) by

$$\langle \nu, \varphi \rangle = \int_\Omega \langle \nu_x, f(x, \cdot) \rangle \, dx \qquad \text{for} \quad f(x, y) \in C_c(\Omega \times [-m, m])$$

which is the space of continuous, compactly supported, real functions on $\Omega \times [-m, m]$. Moreover, for $f(x) \in C_c(\Omega)$, we have

$$\langle \nu, f \rangle = \int_\Omega f(x) \, dx$$

that is to say, the projection of ν on Ω is dx.

Applying the desintegration theorem of Jirina,[17] we see that this gives an equivalent definition of Young measures. That is, for any positive Radon measure ν on $\Omega \times [-m, m]$ whose projection on Ω is dx, there is a measurable mapping $x \rightsquigarrow \nu_x$ satisfying

$$\langle \nu, f \rangle = \int_\Omega \langle \nu_x, f(x, \cdot) \rangle \, dx \qquad \text{for} \quad f \in C_c(\Omega \times [-m, m])$$

The mapping $x \rightsquigarrow \nu_x$ is unique up to the dx-almost everywhere equality.

To any measurable function $\omega: \Omega \to [-m, m]$, we associate the Young measure $\delta_\omega: x \to \delta_{\omega(x)}$, Dirac mass at $\omega(x)$. We shall denote by M the convex set of Young measures on $\Omega \times [-m, m]$, and we recall some useful properties.

(i) M is closed in the space $M_b(\Omega \times [-m, m])$ of all bounded Radon measures on $\Omega \times [-m, m]$ (with the narrow topology), the narrow topology is equal on M to the vague topology (associated to the continuous compactly supported function), and this topology is metrizable. Furthermore, as $[-m, m]$ is compact, M is compact.

(ii) By the mapping $\omega \rightsquigarrow \delta_\omega$, we identify $L_m^\infty(\Omega)$ to a dense subset of M.

To any Young measure ν, we can associate a mean function $\bar{\nu}$ by $\bar{\nu}(x) = \int y \, d\nu_x$; of course, $\bar{\nu} \in L_m^\infty(\Omega)$, and the mapping $\nu \rightsquigarrow \bar{\nu}$ is continuous from the narrow to the weak $*$-topology.

Now, for a given initial data ω_0, we consider the Young measures $\delta_{\omega(t)}$. Since M is compact and metrizable, we can find sequences $t_n \to +\infty$ such that $\delta_{\omega(t_n)}$ converges narrowly toward some Young measure ν. It follows that $\omega(t_n)$ converges weakly toward $\bar{\nu}$ and $E(\omega(t_n)) \to E(\bar{\nu})$, which is the energy of the Young measure ν (notice that it depends only on the mean value and that the microscopic oscillations have no energy).

Now the narrow convergence implies that for any continuous function f on $[-m, m]$, we have

$$\int_\Omega f(\omega(t_n))\,dx \to \int_\Omega \langle \nu_x, f\rangle\,dx$$

and from $\int_\Omega f(\omega(t_n))\,dx = \langle \pi_{\omega_0}, f\rangle$, we get $\pi_{\omega_0} = \int_\Omega \nu_x\,dx$, and we shall say that ν is a mixture of ω_0. So, we see that the limit ν contains the whole information given by the constants of the motion.

Let us now describe how the Eulerian flow extends to the extended phase space M.

Proposition 2. There is a unique semigroup $\bar{\Phi}_t$ of homeomorphisms of M extending Φ_t, that is,

$$\bar{\Phi}_t(\delta_\omega) = \delta_{\Phi_t(\omega)} \qquad \text{for all } \omega \text{ in } L_m^\infty(\Omega)$$

Proof. Let us define $\bar{\Phi}_t$ on M by $\bar{\Phi}_t(\nu)_x = \nu_{\varphi_t^{-1}(x)}$, where φ_t is the Lagrangian flow associated to the initial vorticity $\bar{\nu}$. As Φ_t is a semigroup, then $\bar{\Phi}_t$ is a semigroup, too. We easily check that $\bar{\Phi}_t$ is one-to-one. And now we prove that $\bar{\Phi}_t$ is continuous on M.

Let ν^n be a sequence converging toward ν and $f(x, y) \in C_c(\Omega \times [-m, m])$; we have to prove that

$$\int_\Omega \langle \bar{\Phi}_t(\nu^n)_x, f(x, \cdot)\rangle\,dx \to \int_\Omega \langle \bar{\Phi}_t(\nu)_x, f(x, \cdot)\rangle\,dx$$

And by change of variables

$$\int_\Omega \langle \nu^n_{x'}, f(\varphi_t^n(x'), \cdot)\rangle\,dx' \to \int_\Omega \langle \nu_{x'}, f(\varphi_t(x'), \cdot)\rangle\,dx'$$

We write the first term:

$$\int_\Omega \langle \nu^n_x, f(\varphi_t^n(x), \cdot) - f(\varphi_t(x)), \cdot)\rangle\,dx + \int_\Omega \langle \nu^n_x, f(\varphi_t(x), \cdot)\,dx$$

and the result follows from the uniform convergence on Ω of φ_t^n toward φ_t (as $\bar{\nu}^n$ converges weakly toward $\bar{\nu}$; see ref. 10). The uniqueness of $\bar{\Phi}_t$ follows from the density of $L_m^\infty(\Omega)$ in M. ∎

Remark. Heuristically, we can say that the microscopic oscillations are merely frozen and transported by the mean velocity.

For the sequel, the space $L^\infty(\Omega)$ will be the space of microscopic states, and the set of Young measures with bounded support the set of macroscopic states.

4. CONCENTRATION AND ENTROPY FOR YOUNG MEASURES

All the sequel is based on the following fact, which we present first heuristically.

Let us suppose that we "randomly" choose a microstate ω in the subset of the functions of $L_m^\infty(\Omega)$ satisfying $E(\omega)=E(\omega_0)$ and $\pi_\omega=\pi_{\omega_0}$. Then "with a high probability" δ_ω will be very close (in the narrow topology) to a well-determined Young measure ν^*. In other words, to a great majority of the microstates in the subset given above, we can associate a unique macrostate ν^*. Of course this presentation is oversimplified and to have a general result, ν^* has to be replaced by a set $\mathscr{E}^*$ of macrostates. We find the set $\mathscr{E}^*$ by maximizing a convenient entropy functional.

To give a precise meaning to these considerations, we now introduce some technicalities. We may consider the following definition and concentration theorem as a convenient reformulation of well-known results from large-deviation theory.[5,13]

Let π_0 be a given probability measure on $[-m, m]$ and $\pi = dx \otimes \pi_0$ the associated Young measure. Then we define a notion of concentration by means of piecewise constant functions. Let $\mathscr{X}$ be an equipartition of Ω, that is, $\mathscr{X}=\{X^i \mid i=1\cdots n(\mathscr{X})\}$ is a finite measurable partition of Ω such that $|X^i|=|X^j|$ for all i, j. Define the diameter of $\mathscr{X}$:

$$\delta(\mathscr{X})=\sup_i \sup\{|x-x'| \mid x, x' \in X^i\}$$

Given $y_1,\dots, y_n \in [-m, m]^n$, we denote $\check{\mathscr{X}}(y_1,\dots, y_n)$ the Young measure associated to the measurable step function equal to y_i on the set X^i. On $[-m, m]^n$ we put the probability measure $\bigotimes^n \pi_0$ and then consider $\check{\mathscr{X}}$ as a random variable taking its values in M (notice that $\check{\mathscr{X}}$ is continuous from $[-m, m]^n$ to M).

Definition. Let $\mathscr{E}, \mathscr{E}^*$ be subsets of M. We say that $\mathscr{E}^*$ concentrates conditionally to $\mathscr{E}$ iff:

(i) $\forall W'$:

$$\liminf_{\delta(\mathscr{X})\to 0} \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{W'}) > -\infty$$

(ii) $\forall W^*$, $\exists \alpha > 0$, $\exists W$, $\forall W'$, $\exists \eta > 0$, $\forall \mathscr{X}$ such that $\delta(\mathscr{X}) \leqslant \eta$:

$$\frac{\text{Prob}(\check{\mathscr{X}} \in \mathscr{E}_W \backslash \mathscr{E}^*_{W^*})}{\text{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{W'})} \leqslant e^{-n(\mathscr{X})\alpha}$$

Here W^*, W, W' denote open neighborhoods of 0 (for the narrow topology) in $M_b(\Omega \times [-m, m])$ and $\mathscr{E}_W = (\mathscr{E} + W) \cap M$.

This rather sophisticated definition needs some comments.

Remarks:

1. Heuristically, it means that a great majority of piecewice constant measurable functions which are in a neighborhood of $\mathscr{E}$ are in fact in a neighborhood of $\mathscr{E}^*$.

2. To have a consistent definition, we have to widen the sets $\mathscr{E}$, $\mathscr{E}^*$ into open neighborhoods. Notice that $\text{Prob}(\check{\mathscr{X}} \in \mathscr{E})$ is not defined for an arbitrary set $\mathscr{E}$; and even if $\mathscr{E}$ is a Borel subset, it can be zero.

3. The assumption (i) ensures that when $\delta(\mathscr{X}) \to 0$, $\text{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{W'})$ cannot be too small.

From a previous work, we know the entropy functional which enables us to find the concentration sets $\mathscr{E}^*$.[32,33] It is the Kullback entropy, given on M by

$$K_\pi(\nu) = -\int \text{Log} \frac{d\nu}{d\pi} \, d\nu$$

if ν is absolutely continuous with respect to π

$$K_\pi(\nu) = -\infty \qquad \text{otherwise}$$

And we can now give the following result:

Concentration Theorem 3.[32,33] Let $\mathscr{E}$ be a closed, nonempty subset of M, and $\mathscr{E}^*$ the subset of $\mathscr{E}$ where the functional K_π achieves its maximum value on $\mathscr{E}$. Then $\mathscr{E}^*$ concentrates conditionally to $\mathscr{E}$.

Remarks:

1. It is well known[5,13] that K_π is a concave upper semicontinuous functional. Moreover, K_π is strictly concave on the domain $\{\nu \mid K_\pi(\nu) > -\infty\}$, and sup-compact; that is to say, for any real number b the set $\{\nu \mid K_\pi(\nu) \geqslant b\}$ is a compact (convex) subset of M. So, we deduce that $\mathscr{E}^*$ is a nonempty closed set.

2. The concentration property given here is slightly stronger than the one we define in refs. 32 and 33. Nevertheless, one can easily check that the proof given in ref. 33 actually gives this stronger result.

3. Our notion of concentration depends on the choice of a basic probability measure π_0 on $[-m, m]$. This choice of π_0 gives the suitable entropy functional (K_π with $\pi = dx \otimes \pi_0$) for the problem at hand. For example, when we work with mixtures of a given vorticity function ω, we obviously take $\pi_0 = (1/|\Omega|)\pi_\omega$.

4. The case where $\mathscr{E}^*$ is not reduced to a point corresponds to a phase transition situation.

5. Let ω: $\Omega \to [-m, m]$ be a measurable function and $\nu \in M$ a mixture of ω. We can say that ν performs a mixing of the values of ω while they are globally preserved; for example, $\pi = dx \otimes (1/|\Omega|)\pi_\omega$ is a mixture of ω.

Let us suppose now that ω takes only n distinct values $a_1,\ldots,a_n$, and denote $\Omega^i = \{x \in \Omega \mid \omega(x) = a_i\}$.

We have

$$\pi_\omega = \sum_{i=1}^{n} |\Omega^i| \, \delta_{a_i}$$

If ν is a mixture of ω, one readily sees that there are n nonnegative measurable functions $e_i(x)$ satisfying

$$\sum_{i=1}^{n} e_i(x) = 1 \qquad \text{and} \qquad \int_\Omega e_i(x)\, dx = |\Omega^i|$$

such that

$$\nu_x = e_1(x)\delta_{a_1} + \cdots + e_n(x)\delta_{a_n}$$

for almost all x in Ω.

A straightforward computation gives for the Kullback entropy

$$K_\pi(\nu) = -\int_\Omega \sum_i e_i(x) \operatorname{Log} e_i(x)\, dx + \sum_i |\Omega^i| \operatorname{Log} |\Omega^i|$$

with the convention $0 \operatorname{Log} 0 = 0$.

We see that up to an additive constant, $K_\pi(\nu)$ is equal to the classical Boltzmann mixing entropy.

We come now to a property which is essential in our program; that is, the conservation of the concentration property by the extended Eulerian flow on M.

Proposition 4. If $\mathscr{E}^*$ concentrates conditionally to $\mathscr{E}$, then, for all t, $\bar{\Phi}_t(\mathscr{E}^*)$ concentrates conditionally to $\bar{\Phi}_t(\mathscr{E})$.

Proof. Let us suppose that $\mathscr{E}^*$ concentrates conditionally to $\mathscr{E}$. We know that $\bar{\Phi}_t$ is a homeomorphism and M is compact. Then $\bar{\Phi}_t$ and $\bar{\Phi}_t^{-1}$ are uniformly continuous and it suffices to prove: $\forall W^*$, $\exists \alpha > 0$, $\exists W$, $\forall W'$, $\exists \eta$, $\forall \mathscr{X}$ such that $\delta(\mathscr{X}) \leqslant \eta$,

$$\frac{\operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_W \backslash \mathscr{E}^*_{W^*}))}{\operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_{W'}))} \leqslant e^{-n(\mathscr{X})\alpha}$$

To prove this point, we make use of an extended form of the Cramer–Chernoff inequalities.[23]

Proposition 5. Let A be a Borel subset of M; then we have

$$\limsup_{\delta(\mathscr{X}) \to 0} \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in A) \leqslant \sup_{v \in \bar{A}} K_\pi(v)$$

$$\liminf_{\delta(\mathscr{X}) \to 0} \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in A) \geqslant \sup_{v \in \mathring{A}} K_\pi(v)$$

From this proposition, we get

$$\begin{aligned}\limsup \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_W \backslash \mathscr{E}^*_{W^*})) &\leqslant \sup_{v \in \bar{\Phi}_t(\overline{\mathscr{E}_W \backslash \mathscr{E}^*_{W^*}})} K_\pi(v) \\ &\leqslant \sup_{v \in \bar{\Phi}_t(\mathscr{E}_{2W} \backslash \bar{\mathscr{E}}^*_{(1/2)W^*})} K_\pi(v)\end{aligned}$$

Now we easily verify that K_π is conserved by $\bar{\Phi}_t$ [i.e., $K_\pi(\bar{\Phi}_t(v)) = K_\pi(v)$, for all v], and then

$$\begin{aligned}\sup_{v \in \bar{\Phi}_t(\mathscr{E}_{2W} \backslash \bar{\mathscr{E}}^*_{(1/2)W^*})} K_\pi(v) &= \sup_{v \in \mathscr{E}_{2W} \backslash \bar{\mathscr{E}}^*_{(1/2)W^*}} K_\pi(v) \\ &\leqslant \liminf \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{2W} \backslash \mathscr{E}^*_{(1/2)W^*})\end{aligned}$$

So, we have

$$\begin{aligned}&\limsup \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_W \backslash \mathscr{E}^*_{W^*})) \\ &\quad \leqslant \liminf \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{2W} \backslash \mathscr{E}^*_{(1/2)W^*})\end{aligned}$$

A similar computation gives

$$\liminf \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_{W'})) \geqslant \limsup \frac{1}{n(\mathscr{X})} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{(1/2)W'})$$

This, of course, implies (i). And we have

$$\liminf \frac{1}{n} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_{W'})) - \limsup \frac{1}{n} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \bar{\Phi}_t(\mathscr{E}_W \backslash \mathscr{E}^*_{W^*}))$$

$$\geqslant \limsup \frac{1}{n} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{(1/2)W'}) - \liminf \frac{1}{n} \operatorname{Log} \operatorname{Prob}(\check{\mathscr{X}} \in \mathscr{E}_{2W} \backslash \mathscr{E}^*_{(1/2)W^*})$$

and this last term is $\geqslant \alpha$ by the hypothesis that $\mathscr{E}^*$ concentrates conditionally to $\mathscr{E}$. The result readily follows. ▮

5. EQUILIBRIUM STATES

Suppose given an initial datum ω_0 in $L^\infty(\Omega)$. It follows from the concentration theorem that the microstates ω satisfying $\pi_\omega = \pi_{\omega_0}$ and $E(\omega) = E(\omega_0)$ are concentrated about the set $\mathscr{E}^*$ of the macrostates v^* which are solutions of the variational problem:

$$\text{(V.P.)} \quad K_\pi(v^*) = \max_{v \in \mathscr{E}} K_\pi(v)$$

where the set $\mathscr{E}$ is

$$\mathscr{E} = \left\{ v \in M \mid E(\bar{v}) = E(\omega_0) \text{ and } \int_\Omega v_x \, dx = \pi_{\omega_0} \right\}$$

and $\pi = dx \otimes \pi_0$, where $\pi_0 = (1/|\Omega|)\pi_{\omega_0}$.

One easily checks that the set $\mathscr{E}$ is closed (this is a straightforward consequence of the complete continuity of the mapping $\omega \rightsquigarrow \psi$). And we have seen that $\mathscr{E}^*$ is nonempty and closed.

The set $\mathscr{E}$ is obviously conserved by the flow [i.e., $\bar{\Phi}_t(\mathscr{E}) = \mathscr{E}$], and since K_π is also conserved by $\bar{\Phi}_t$, we have

$$\bar{\Phi}_t(\mathscr{E}^*) = \mathscr{E}^*$$

We shall call the set $\mathscr{E}^*_m = \{\bar{v}^* \mid v^* \in \mathscr{E}^*\}$ the equilibrium set corresponding to the initial datum ω_0. Of course, we have $\Phi_t(\mathscr{E}^*_m) = \mathscr{E}^*_m$.

In the particular case where $\mathscr{E}^* = \{v^*\}$, this implies that $\bar{v}^*$ is a stationary solution of the Euler equations, the equilibrium state.

Now, to proceed further in the determination of the equilibrium set, we have to solve (V.P.).

First, we write down the equation of the Gibbs states which is the equation satisfied by the critical points of the functional K_π on the set $\mathscr{E}$.

Let us assume that v^* is a solution of (V.P.) such that $K_\pi(v^*) > -\infty$. Then $v^* = \rho^*(x, y)\pi$, where $\rho^* \in L^1(\pi)$. Furthermore, we shall assume that $C_1 \geqslant \rho^*(x, y) \geqslant C_0 > 0$, π-a.e.

For $\rho \in L^1(\pi)$, $\rho \geqslant 0$, let us denote

$$S(\rho) = K_\pi(\rho\pi) = -\int \rho \operatorname{Log} \rho \, d\pi$$

Then, $S(\rho^*)$ is the maximum value of $S(\rho)$ on the linear submanifold of $L^1(\pi)$ defined by

$$\int \rho(x, y) \, d\pi_0(y) = 1, \qquad dx\text{-a.e.}$$

$$\int \rho(x, y) \, dx = |\Omega|, \qquad \pi_0\text{-a.e.}$$

and satisfying the nonlinear energy constraint

$$E(\rho) = E\left(\int y\rho(x, y) \, d\pi_0(y)\right) = E(\omega_0)$$

Then, it is easy to check that the functionals $S(\rho)$ and $E(\rho)$ are continuously differentiable in a neighborhood of ρ^* in the space $L^\infty(\pi)$. And we have, for $\delta\rho \in L^\infty(\pi)$,

$$\delta S = -\int (1 + \operatorname{Log} \rho^*) \delta\rho \, d\pi(x, y)$$

$$\delta E = +\int \psi^*(x) \, y \, \delta\rho \, d\pi(x, y)$$

where ψ^* is the stream function associated to

$$\omega^*(x) = \int y\rho^*(x, y) \, d\pi_0(y)$$

Application of the Lagrange multiplier rule gives the existence of a parameter β such that $\delta S = \beta \delta E$ for any variation $\delta\rho \in L^\infty(\pi)$ satisfying

$$\int \delta\rho(x, y) \, d\pi_0(y) = 0, \qquad dx\text{-a.e.}$$

and

$$\int \delta\rho(x, y) \, dx = 0, \qquad \pi_0\text{-a.e.}$$

Tedious but classical computations then give

$$\rho^*(x, y) = \frac{1}{Z(x)} e^{-\alpha(y) - \beta y \psi^*(x)}$$

where $\alpha(y)$ is some continuous function, and

$$Z(x) = \int e^{-\alpha(y) - \beta y \psi^*(x)} \, d\pi_0(y)$$

So, we see that the stream function ψ^* necessarily satisfies the following equation of Gibbs states:

$$\text{(G.S.E.)} \quad \begin{cases} -\Delta\psi = \dfrac{-1}{\beta} \dfrac{d}{d\psi} \operatorname{Log} Z \\ \psi = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where $Z(\psi) = z(-\beta\psi)$, and we denote

$$z(\xi) = \int e^{-\alpha(y) + \xi y} \, d\pi_0(y)$$

The function $\operatorname{Log} z(\xi)$ is strictly convex; indeed, we have

$$\frac{d^2}{d\xi^2} \operatorname{Log} z = \frac{1}{z^2} [zz'' - (z')^2]$$

and this last term is strictly positive by application of the Cauchy–Schwarz inequality:

$$\left(\int e^{[-\alpha(y) + \xi y]/2} e^{[-\alpha(y) + \xi y]/2} y \, d\pi_0 \right)^2 < \left(\int e^{-\alpha(y) + \xi y} \, d\pi_0 \right) \left(\int e^{-\alpha(y) + \xi y} y^2 \, d\pi_0 \right)$$

Then we get the striking result that for a Gibbs state the function $\omega = f(\psi)$ is either strictly decreasing (case $\beta > 0$), strictly increasing (case $\beta < 0$), or perhaps a constant (if $\beta = 0$). Furthermore, we have

$$\frac{d^2}{d\xi^2} \operatorname{Log} z = \frac{z''}{z} - \left(\frac{z'}{z} \right)^2 \leqslant \frac{z''}{z} \leqslant \max\{ y^2 \mid y \in \operatorname{Supp} \pi_0 \}$$

and we have the following existence-uniqueness result for (G.S.E.).

Proposition 6. For $-\beta \max\{y^2 \mid y \in \operatorname{Supp} \pi_0\} < \lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ (associated to the Dirichlet boundary value condition), the equation (G.S.E.) has a unique solution ψ^* in the Sobolev space $H_0^1(\Omega)$.

Proof. As the right-hand side of (G.S.E.) is a continuous and bounded function of ψ, the existence of a solution follows in a standard way by Schauder's fixed point theorem (of course for all values of β).

Uniqueness for β satisfying the above condition follows from the fact that the solutions of (G.S.E.) are the critical points of the functional

$$F(\psi) = \frac{1}{2}\int_\Omega (\nabla\psi)^2\,dx + \frac{1}{\beta}\int_\Omega \operatorname{Log} Z(\psi)\,dx, \qquad \text{defined for } \psi \in H_0^1(\Omega)$$

A straightforward computation gives for the second variation of the functional F

$$\delta^2 F = \frac{1}{2}\int_\Omega (\nabla\delta\psi)^2\,dx + \frac{\beta}{2}\int_\Omega \frac{d^2}{d\xi^2}\operatorname{Log} z(-\beta\psi)(\delta\psi)^2\,dx$$

The classical inequality

$$\int_\Omega (\nabla\delta\psi)^2\,dx \geqslant \lambda_1 \int_\Omega (\delta\psi)^2\,dx$$

then gives

$$\delta^2 F \geqslant \frac{1}{2}\int_\Omega \left[\lambda_1 + \beta\frac{d^2}{d\xi^2}\operatorname{Log} z(-\beta\psi)\right](\delta\psi)^2\,dx$$

For $\beta \geqslant 0$, we have

$$\delta^2 F \geqslant \frac{\lambda_1}{2}\int_\Omega (\delta\psi)^2\,dx$$

and for $\beta < 0$

$$\delta^2 F \geqslant \frac{1}{2}(\lambda_1 + \beta \max\{y^2 \mid y \in \operatorname{supp} \pi_0\})\int_\Omega (\delta\psi)^2\,dx$$

Thus, the functional F is strictly convex on the space $H_0^1(\Omega)$, and a critical point ψ^* is necessarily the unique minimum of F. ∎

Now, let us go back to the variational problem. We have seen that if ν^* is a solution of (V.P.) (such that $C_1 \geqslant \rho^* \geqslant C_0 > 0$, π-a.e..), then the corresponding stream function ψ^* satisfies (G.S.E.) for some Lagrange multipliers $\alpha(y)$, β.

Of course, the values of the multipliers $\alpha(y)$, β depend on the values of the constraints

$$E(\bar{v}^*) = E(\omega_0) \tag{1}$$

$$\int_\Omega v_x^* \, dx = \pi_{\omega 0} \tag{2}$$

Conversely, let us choose any continuous function $\alpha(y)$ and a real number β satisfying the condition of Proposition 6. Let $\psi^{\alpha,\beta}$ be the unique solution of the (G.S.E.), $\rho^{\alpha,\beta}$ the corresponding density function:

$$\rho^{\alpha,\beta}(x, y) = \frac{1}{Z} e^{-\alpha(y) - \beta y \psi^{\alpha,\beta}(x)} \qquad \text{and} \qquad v^{\alpha,\beta} = \rho^{\alpha,\beta}\pi$$

Notice that there is no reason for the macrostate $v^{\alpha,\beta}$ to satisfy the constraints (1) and (2), since α, β are arbitrarily chosen. In fact, $v^{\alpha,\beta}$ is the solution of the following variational problem.

Proposition 7. Let $\alpha(y)$, β be given (β satisfying the condition of Proposition 6). Then $v^{\alpha,\beta}$ is solution of the variational problem

$$K_\pi(v^{\alpha,\beta}) = \max_{v \in \mathscr{E}^{\alpha,\beta}} K_\pi(v)$$

where

$$\mathscr{E}^{\alpha,\beta} = \left\{ v \in M \mid E(\bar{v}) = E(\bar{v}^{\alpha,\beta}) \text{ and } \int_\Omega v_x \, dx = \int_\Omega v_x^{\alpha,\beta} \, dx \right\}$$

Proof. One easily sees that $\rho^{\alpha,\beta}$ is a critical point of the functional

$$\mathscr{T}(\rho) = S(\rho) - \beta E(\rho) - \int \alpha(y) \, \rho(x, y) \, d\pi$$

on the linear submanifold of $L^\infty(\pi)$ defined by

$$\int \rho(x, y) \, d\pi_0(y) = 1, \qquad dx\text{-a.e.}$$

Then, we distinguish two cases.

For $\beta \geqslant 0$, since $S(\rho)$ is strictly concave and $E(\rho)$ is convex, $\mathscr{T}(\rho)$ is strictly concave and $\rho^{\alpha,\beta}$ is the unique maximum of $\mathscr{T}$.

As a consequence $S(\rho^{\alpha,\beta}) \geqslant S(\rho)$, for all ρ satisfying

$$E(\rho) = E(\rho^{\alpha,\beta}) \qquad \text{and} \qquad \int_\Omega \rho(x, y)\, dx = \int_\Omega \rho^{\alpha,\beta}(x, y)\, dx, \qquad \pi_0\text{-a.e.}$$

The case $\beta < 0$ is less straightforward. First, we compute the second variation of the functional $\mathscr{T}$ on the open convex set of $L^\infty(\pi)$:

$$U = \{\rho \in L^\infty(\pi) \mid \exists \varepsilon > 0, \rho \geqslant \varepsilon\}$$

We get

$$\delta^2 \mathscr{T} = -\frac{1}{2} \int \frac{(\delta\rho)^2}{\rho}\, d\pi - \frac{\beta}{2} \int_\Omega \delta\psi\, \delta\omega\, dx$$

where $\delta\omega(x) = \int y\, \delta\rho(x, y)\, d\pi_0(y)$, and $\delta\psi$ is the corresponding stream function.

Then, using the classical inequality

$$\int_\Omega \delta\psi\, \delta\omega\, dx \leqslant \frac{1}{\lambda_1} \int_\Omega (\delta\omega)^2\, dx$$

we obtain

$$\delta^2 \mathscr{T} \leqslant -\frac{1}{2} \int \frac{(\delta\rho)^2}{\rho}\, d\pi - \frac{\beta}{2\lambda_1} \int_\Omega (\delta\omega)^2\, dx$$

Furthermore, we have

$$(\delta\omega)^2 \leqslant \max\{y^2 \mid y \in \operatorname{supp} \pi_0\} \left[\int |\delta\rho|\, d\pi_0(y)\right]^2$$

and the Cauchy–Schwarz inequality yields

$$\left[\int |\delta\rho|\, d\pi_0(y)\right]^2 \leqslant \int \frac{(\delta\rho)^2}{\rho}\, d\pi_0(y)$$

from which

$$\delta^2 \mathscr{T} \leqslant -\frac{1}{2}\left(1 + \frac{\beta}{\lambda_1} \max\{y^2 \mid y \in \operatorname{supp} \pi_0\}\right) \int_\Omega \left[\int |\delta\rho|\, d\pi_0(y)\right]^2 dx$$

Then we see that $\delta^2 \mathscr{T} < 0$ for all $\delta\rho \neq 0$, and the functional $\mathscr{T}$ is strictly concave. Thus, the critical point $\rho^{\alpha,\beta}$ is the unique maximum of $\mathscr{T}$. We conclude, as above, that it gives a maximum entropy state. ∎

Remark. We define

$$k(y) = \frac{1}{|\Omega|} \int_\Omega \rho^{\alpha,\beta}(x, y)\, dx$$

$k(y)$ is a strictly positive continuous function. Let us denote $\tilde{\pi}_0 = k(y)\pi_0$, and $\tilde{\pi} = dx \otimes \tilde{\pi}_0$.

For any mixture ν of $|\Omega|\,\tilde{\pi}_0$, we have

$$K_{\tilde{\pi}}(\nu) = K_\pi(\nu) - |\Omega| \int k(y) \operatorname{Log} k(y)\, d\pi_0(y)$$

Thus $\nu^{\alpha,\beta}$ gives also the maximum value of $K_{\tilde{\pi}}(\nu)$ among the mixtures of $|\Omega|\,\tilde{\pi}_0$ with energy $E(\bar{\nu}^{\alpha,\beta})$.

6. COMMENTS

1. Of course, any solution of the (G.S.E.) is a stationary solution of the Euler equations. We may wonder about the stability of these solutions. It is not hard to see that under the condition of Proposition 6, Arnold's classical estimate[4] applies and gives the stability of the solution in the enstrophy norm:

Let ψ^* be the unique solution of (G.S.E.) and ω^* the corresponding vorticity; then for any ω_0 in $L^\infty(\Omega)$, we have

$$\int_\Omega (\Phi_t \omega_0 - \omega^*)^2\, dx \leqslant c \int_\Omega (\omega_0 - \omega^*)^2\, dx, \qquad \text{for all } t$$

Furthermore, we can prove that any mixture ν of ω^* such that $E(\bar{\nu}) = E(\omega^*)$ is equal to δ_{ω^*}. So, if we repeat the process, starting with ω^*, we shall get ω^* again as an equilibrium state.

2. When some supplementary constants of the motion occur, we must take them into account. This is the case for the circulation around the obstacles when Ω is not simply connected, or the angular momentum when Ω is a ball. This leads to some minor changes in the equation of Gibbs states.[35]

3. The description of the set of solutions of (G.S.E.) for all the values of the parameter β is a complicated question which seems to correspond to the diversity of coherent states that we can observe in experiments.[38] In the particular case of a vortex patch, it has been shown[12] that, when $-\beta$ reaches a positive critical value, a bifurcation occurs with the appearance of another branch of solutions (see also ref. 41).

4. As β is the Lagrange multiplier of the energy constraint, it is the inverse of a temperature. And Proposition 6 proves the existence of equilibrium states with a negative temperature (this phenomenon was foreseen by Onsager[29]). We show now how the existence of such states might be relevant to describe the concentration of vorticity into coherent structures.

Let us consider the case where we have only positive vorticity. As we have seen, an equilibrium state is characterized by a relation $\omega = f(\psi)$. Since $-\Delta\psi = \omega$ and $\psi = 0$ on $\partial\Omega$, the stream function ψ is everywhere positive and reaches its maximum value at some point x^* in Ω.

Then for $\beta > 0$, as we have seen, f is strictly decreasing and as ψ decreases from x^* to the wall $\partial\Omega$, ω increases from x^* to the wall. So, we see that, in this case, the vorticity is essentially located at the wall; while for $\beta < 0$, the same argument shows that the vorticity is concentrated at the vortex core (about x^*) (a more detailled discussion, at a physical level, can be found in ref. 35).

5. The motivation for the study of coherent structures in two-dimensional flows comes mainly from geophysical applications. Jupiter's Great Red Spot is the most spectacular example of such a structure.[15,39]

The physical relevance of this theory is presently the subject of experimental investigations[38] and numerical simulations (the results given in ref. 41, in the case of a shear layer instability, show a very good agreement).

6. Of course, our approach is based on some underlying assumption of ergodicity for the flow Φ_t. As we have no natural invariant measure on the phase space $L^\infty(\Omega)$, we cannot speak of ergodicity in the usual sense. A dynamical justification of the method could be given by a result of the following form: For any ω in a dense open set (for the weak $*$-topology) of $L^\infty(\Omega)$, $\Phi_t\omega$ spends almost all its time in any weak $*$-neighborhood of the set $\mathscr{E}_m^*$ previously defined. That is: for all V, neighborhood of 0 in the weak $*$-topology,

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{mes}\{t \in [0, T] \mid \Phi_t\omega \in \mathscr{E}_m^* + V\} = 1$$

7. A natural question is to relate our approach to the traditional Gibbsian method of statistical mechanics.

Let us denote by $P_{\mathscr{X}}$ the probability measure (on the space of Young measures M) which is the image by the mapping $\tilde{\mathscr{X}}$ of the measure $d\pi_0(y_1) \cdots d\pi_0(y_n)$ (with the notations of Section 4). Our method is based on the fact that the family $P_{\mathscr{X}}$ has the large-deviation property[13] with constants $n(\mathscr{X})$ and entropy function $K_\pi(v)$ (cf. Proposition 5). We use this property to prove our concentration result when we restrict to the microstates which satisfy to the constraints given by the constants of the motion. In this sense, we may say that our approach is microcanonical.

Since the first submission of this work, a paper by Miller[24] has appeared which derives the same result by a mean-field approach. We can see that behind Miller's arguments (which work at a more physical level) there is also a large-deviation property in the space of Young measures. In fact, Miller considers, for β fixed, the Gibbs measures:

$$(1/Z) \exp[-n(\mathscr{X})\, \beta E(\check{\mathscr{X}})]\, d\pi_0(y_1) \cdots d\pi_0(y_n)$$

Let us denote by $Q_{\mathscr{X}}$ the probability measures on M, image by $\check{\mathscr{X}}$ of these Gibbs measures. We can see, using Varadhan's theorem on the asymptotics of integrals,[13] that, when β satisfies the condition of Proposition 6, Baldi's large-deviation theorem[6] applies to the family $Q_{\mathscr{X}}$ and ensures a large-deviation property with constants $n(\mathscr{X})$ and entropy function

$$J(v) = K_\pi(v) - \beta E(v) - \sup_{v \in M} [K_\pi(v) - \beta E(v)]$$

Thus, when we restrict attention to the microstates which satisfy the constraint $\pi_\omega = \pi_{\omega 0}$, we get a concentration property about the unique solution v^* of the variational problem:

$$J(v^*) = \max \left\{ J(v) \mid v \in M, \int v_x\, dx = \pi_{\omega 0} \right\}$$

Of course, this problem yields the same (G.S.E.) for the critical points. We see that the difference from our approach lies in handling here the energy constraint in a canonical way. Note, however, that when β does not satisfy the condition of Proposition 6, the functional $J(v)$ might not be strictly convex and Baldi's theorem does not apply. The above justification fails in that case.

Note here that the n-dimensional Euclidean volume on the space of piecewise constant vorticity functions is not conserved by the Euler flow. Thus, the method is not clearly related *a priori* to the dynamics of the system. It is the conservation of the concentration property by the flow which provides some compatibility of the statistics with the dynamical system.

It may seem more natural to carry out the thermodynamic limit in the weak space $L^\infty(\Omega)$. The large-deviation property for the image measures is easier to obtain in this framework, but unfortunately, doing so, we cannot take into account the constraint $\pi_\omega = \pi_{\omega_0}$. The Young measures framework is well suited to take into account the whole set of constraints; furthermore, it provides a convenient heuristical picture of the complex time evolution of the vorticity functions.

Altogether, we believe that Baldi's large-deviation theorem, applied in the Young measure framework, provides a powerful tool to carry out the statistical mechanics of a class of infinite-dimensional Hamiltonian systems. This is now the object of active investigation; see, for example, refs. 23 and 40 for the case of quasigeostrophic models.

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